Lipschitz Maps and Nets in Euclidean Space

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Curtis T. McMullen*

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1 Introduction

In this paper we discuss the following three questions.

1. Given a real-valued function $f \in L^\infty(\mathbb{R}^n)$ with $\inf f(x) > 0$, is there a bi-Lipschitz homeomorphism $\phi : \mathbb{R}^n \to \mathbb{R}^n$ such that the Jacobian determinant $\det D\phi = f$?

2. Given $f \in L^\infty(\mathbb{R}^n)$, is there a Lipschitz or quasiconformal vector field with $\operatorname{div} v = f$?

3. Given a separated net $Y \subset \mathbb{R}^n$, is there a bi-Lipschitz map $\phi : Y \to \mathbb{Z}^n$?

When $n = 1$ all three questions have an easy positive answer. In this paper we show that for $n > 1$ the answer to all three questions is *no*. We also find all three questions have positive solutions if the Lipschitz condition is relaxed to a Hölder condition.

Definitions. A map ϕ is *bi-Lipschitz* if there is a constant K such that

$$\frac{1}{K} < \frac{|\phi(x) - \phi(x')|}{|x - x'|} < K$$

for $x \neq x'$. A set $Y \subset \mathbb{R}^n$ is a *net* if there is an R such that $d(x, Y) < R$ for every $x \in \mathbb{R}^n$; it is *separated* if there is an $\epsilon > 0$ such that $|y - y'| > \epsilon > 0$ for every pair $y \neq y'$ in Y.

History. In 1965, J. Moser showed that any two positive, C^∞ volume forms on a compact manifold with the same total mass are related by a diffeomorphism [Mos]. Extensions of this result to other smoothness classes such as $C^{k,\alpha}$ were given in [Rei1] and [DM]; see also [RY1], [RY2], and [Ye].

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Questions (1) and (2) remained open. Question (3) was posed in Gromov's 1993 book [Gr, p.23], and popularized by Toledo's review [Tol].

Recently counterexamples to (1) and (3) were discovered independently by Burago and Kleiner [BK], and the author. Here we show the linearized question (2) can be settled using a 1962 result of Ornstein (§2). The counterexample to (2) suggests the right type of f to make a counterexample to (1), as we sketch in §3. This $f \in L^\infty$ is similar to the one constructed in [BK], to which we refer for a detailed resolution of (1). In §4 we show questions (1) and (3) are equivalent, completing the discussion of Lipschitz mappings. Finally in §5 we show questions (1-3) have positive answers in the Hölder category.

2 Vector fields

We begin with the infinitesimal form of the problem of constructing a map with prescribed volume distortion. That is, we study the equation

$$\operatorname{div} v = \sum \frac{\partial v_i}{\partial x_i} = f$$

on $\mathbb{R}^n$, where f is a real-valued function and $\operatorname{div} v$ is the divergence of the vector field v. We will show:

Theorem 2.1 *For any $n > 1$ there is an $f \in L^\infty(\mathbb{R}^n)$ which is not the divergence of any Lipschitz, or even quasiconformal, vector field.*

Definitions. Let $D = (\partial/\partial x_i)$; then the matrix of partial derivatives of a vector field v is given by the outer product

$$(Dv)_{ij} = \frac{\partial v_i}{\partial x_j},$$

and $\operatorname{div} v = \operatorname{tr}(Dv)$. Similarly, letting

$$(D^2)_{ij} = \frac{\partial^2}{\partial x_i \partial x_j},$$

we have $\Delta f = \operatorname{tr}(D^2 f)$.

A vector field v is *quasiconformal* if the distribution Sv lies in L^∞, where the *conformal strain*

$$Sv = \frac{1}{2}\left(Dv + (Dv)^*\right) - \frac{1}{n}(\operatorname{tr} Dv)I$$

is the symmetric, trace-free part of Dv. Explicitly,

$$(Sv)_{ij} = \frac{1}{2}\left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i}\right) - \frac{1}{n}\sum_k \frac{\partial v_k}{\partial x_k}.$$

Any Lipschitz vector field is quasiconformal.

Quasiconformal vector fields with $\operatorname{div} v \in L^\infty$ are more general than Lipschitz vector fields, but they provide good models for infinitesimal bi-Lipschitz maps. For example, $v(z) = iz \log|z|$ is not Lipschitz, but it generates a Lipschitz isotopy of the plane (shearing along circles). Theorem 2.1 states that even this broader class of quasiconformal vector fields is insufficient to solve $\operatorname{div} v = f$. (Further discussion of quasiconformal flows can be found in [Rei2] and [Mc2, Appendix A].)

Singular integral operators. Before proving Theorem 2.1, we mention how it fits into the general theory of singular integral operators and PDE.

Suppose $f \in C_0^\infty(\mathbb{R}^n)$ and $\int f = 0$. The most straightforward solution to $\operatorname{div} v = f$ is given by $v = Du$, the gradient of the solution to Laplace's equation $\Delta u = f$. The regularity of v is thus determined by the behavior of the operator

$$Tf = Dv = D^2 \Delta^{-1} f.$$

For example v is Lipschitz iff $Dv = Tf \in L^\infty$.

The operator T is a singular integral operator of Calderón-Zygmund type, whose kernel is obtained by differentiating a fundamental solution to Laplace's equation. By the general theory of such operators, T sends L^p into L^p for $1 < p < \infty$, but it does not preserve L^∞ or L^1.

In the case at hand, where f is in L^∞, one can say at most that $Dv = Tf \in BMO$ with

$$\|Dv\|_{BMO} \le C_n \|f\|_\infty$$

(see [St, IV.4.1]). Just as vector fields with $Dv \in L^\infty$ are Lipschitz, those with $Dv \in BMO$ satisfy the Zygmund condition

$$||v||_Z \quad = \quad \sup_{x,y \in \mathbb{R}^n, y \neq 0} \frac{|v(x+y) + v(x-y) - 2v(x)|}{|y|} \quad < \quad \infty$$

(see [Mc2, Thm. A.2]). It follows that v has an $|x \log x|$ modulus of continuity, so while v is generally not Lipschitz it is Hölder of every exponent $\alpha < 1$.

On the other hand, a solution to $\operatorname{div} v = f$ is only determined up to a volume-preserving vector field w, so another solution $v + w$ might be Lipschitz even if v is not.

To handle the kernel of the divergence operator, one is lead to argue by duality. Theorem 2.1 then reduces to a problem in L^1, which is settled by the following:

Theorem 2.2 (Ornstein) *For any set of linearly independent degree m differential operators on $\mathbb{R}^n$,*

$$P_i = \sum_{|\alpha|=m} a_i^\alpha \frac{\partial^\alpha}{\partial x^\alpha}, \quad i = 0, \ldots, k,$$

and any $C > 0$, there exists an $g \in C_0^\infty(\mathbb{R}^n)$ such that

$$\|P_0 g\|_1 > C \sum_1^k \|P_i g\|_1.$$

See [Or]; we are grateful to E. Stein for this reference.

Proof of Theorem 2.1. The proof is by contradiction.

Suppose for every $f \in L^\infty(\mathbb{R}^n)$ there exists a quasiconformal vector field v such that $\operatorname{div} v = f$. Then there is a constant C_n such that v can be chosen with

$$\|Sv\|_\infty \quad \leq \quad C_n \|f\|_\infty. \tag{2.1}$$

Indeed, let B be the Banach space of quasiconformal vector fields with bounded divergence, equipped with the (pseudo-)norm

$$\|v\|_B \quad = \quad \|Sv\|_\infty + \|\operatorname{div} v\|_\infty;$$

then the divergence map $\operatorname{div} : B \to L^\infty(\mathbb{R}^n)$ is surjective, so (2.1) follows by the open mapping theorem.

We claim (2.1) implies, for any compactly supported smooth function g, that

$$\|\Delta g\|_1 \quad \leq \quad \frac{n}{n-1} C_n \|Eg\|_1.$$

Here E denotes the trace-zero part of D^2; it satisfies

$$(D^2 g)_{ij} = (Eg)_{ij} + \frac{1}{n}(\Delta g) I_{ij}, \tag{2.2}$$

where $I_{ij} = \delta_{ij}$ is the identity matrix.

The main point of the proof is the identity:

$$\operatorname{tr}(E(Sv)) = \sum E_{ij}(Sv)_{ji} = \frac{n-1}{n} \Delta \operatorname{div} v. \tag{2.3}$$

To check (2.3), note that

$$\operatorname{tr}((D^2)(Dv)) = \sum_{i,j} \frac{\partial^3 v_i}{\partial x_j^2\, \partial x_i} = \Delta \operatorname{div} v,$$

while

$$\frac{1}{n} \operatorname{tr}((\Delta I)(Dv)) = \frac{1}{n} \Delta \operatorname{div} v;$$

so by (2.2) we have

$$\operatorname{tr}(E(Dv)) = \frac{n-1}{n} \Delta \operatorname{div} v.$$

But E is trace-zero and symmetric, so $\operatorname{tr}(E(Dv)) = \operatorname{tr}(E(SV))$ and we have (2.3).

Now given any $g \in C_0^\infty(\mathbb{R}^n)$, choose $f \in L^\infty$ such that $|f| = 1$ and

$$\|\Delta g\|_1 = \int f \Delta g = \int g \Delta f.$$

Choose a quasiconformal vector field with $\operatorname{div} v = f$ and satisfying (2.1), so $\|Sv\|_\infty \leq C_n$. Then

$$\|\Delta g\|_1 = \int g \Delta \operatorname{div} v = \frac{n}{n-1} \int g \operatorname{tr}(E(Sv))$$

by (2.3). Integrating by parts gives

$$\int g \operatorname{tr}(E(Sv)) = \int \operatorname{tr}((Eg)(Sv)),$$

so we have

$$\|\Delta g\|_1 \leq \frac{n}{n-1} \|Eg\|_1 \|Sv\|_\infty \quad \leq \quad \frac{n}{n-1} C_n \|Eg\|_1.$$

But E and Δ are linearly independent differential operators, so this inequality contradicts Ornstein's theorem. ■

3 Maps

In this section we sketch the construction of a counterexample to (1). A similar counterexample is given in [BK, Theorem 1.2]. The L^1 counterexamples given by Ornstein in [Or] are also similar in spirit.

For simplicity we will work in $\mathbb{R}^2$. Let $T \subset S$ denote the square of side $1/3$ within the unit square S. Choose $f > 0$ to be constant on T and $S - T$, with $\int_S f = 1$ and $\int_T f = 0.99$. Cover the edges of S and T with much smaller squares S_i, and redefine $f|S_i$ as $f \circ h_i$, where $h_i : S_i \to S$ is a linear map. See Figure 1; the regions where $f > 1$ are black.

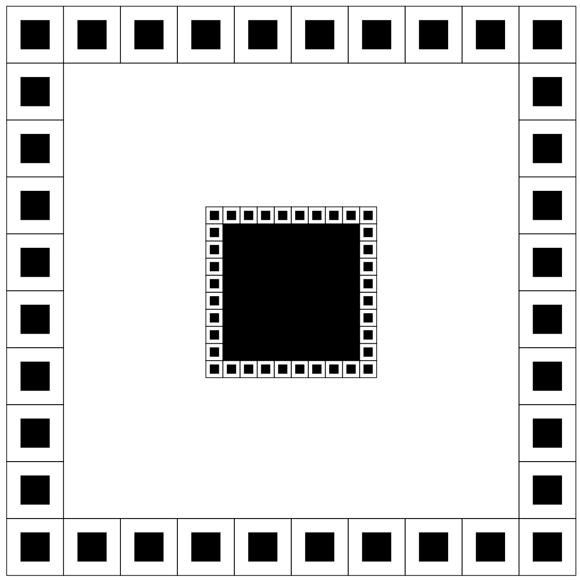

Figure 1. Non-realizable density.

Now repeat the construction along the edges of each S_i, and iterate j times to obtain f_j. As the construction is iterated, arrange that the ratio between the sizes of the squares at levels j and $j+1$ tends to infinity. Then $f(x) = \lim_{j\to\infty} f_j(x)$ exists almost everywhere and is bounded above and below.

We claim f cannot be realized as the Jacobian determinant of a bi-Lipschitz homeomorphism. To see this, let $K = \sup |\phi(a) - \phi(b)|/|a-b|$, where the sup is over *just the edges* $[a, b]$ of all squares at all levels j. For simplicity, suppose K is achieved on a horizontal edge $[a, b]$ of a square S' at level j. Let S_i' denote the squares at level $j+1$ running along $[a, b]$, and let $R = \bigcup S_i'$ be the long, thin rectangular they form.

By the triangle inequality, the horizontal edges of R are mapped to almost straight lines stretched by K. Since area $\phi(R) = \text{area}(R)$, the height of R is compressed by $1/K$. The horizontal edges of most S_i' are also stretched by K, so the perimeter of some S_i' is increased by a factor of at least $K/2$.

But most of the area of $\phi(S_i')$ is filled by $\phi(T_i')$, the image of the black sub-square $T_i' \subset S_i'$. Since the perimeter of T_i' is $1/3$ that of S_i', it is stretched by a factor of about $3K/2$ under ϕ, contradicting the definition of K.

A detailed proof can be given along lines similar to those presented in [BK], to which the reader is referred for a more complete discussion.

This counterexample to (1) was motivated for us by the area-modulus inequality

$$\operatorname{area}(T) \leq \frac{\operatorname{area}(S)}{1 + 4\pi \operatorname{mod}(A)} \tag{3.1}$$

where A is the annulus between two disks $T \subset S \subset \mathbb{C}$ [Mc1, Lemma 2.17]. This inequality relates conformal distortion to distortion of relative areas. Since (3.1) comes from the isoperimetric inequality, for a rigorous proof one is lead to consider stretching along the edges and stability of geodesics as above.

4 Nets

In this section we show questions (1) and (3) are equivalent. In particular, a counterexample to (1) implies a counterexample to (3).

Theorem 4.1 *The following two statements are equivalent:*

A. Every measurable $f > 0$ on $\mathbb{R}^n$ with f and $1/f$ bounded can be realized as the Jacobian determinant of a bi-Lipschitz map.

B. Every separated net $Y \subset \mathbb{R}^n$ is bi-Lipschitz to $\mathbb{Z}^n$.

Proof of Theorem 4.1. (B) $\implies$ (A). Choose a net Y such that under rescaling, the measure that assigns a δ-mass to each point of Y accumulates weakly on the measure $\mu = f(x)\,dx$. By (B) there is a bi-Lipschitz map $\phi : Y \to \mathbb{Z}^n$. Under suitable rescaling, ϕ converges to a bi-Lipschitz homeomorphism $\Phi : \mathbb{R}^n \to \mathbb{R}^n$ with Jacobian f. Compare [BK, Lemma 2.1].

(A) $\implies$ (B). Let $Y \subset \mathbb{R}^n$ be a separated net. Let $\langle C_y \,:\, y \in Y \rangle$ be the tiling of $\mathbb{R}^n$ determined by the Voronoi cells

$$C_y = \{x \,:\, |x - y| < |x - y'| \text{ for all } y' \neq y \text{ in } Y\}.$$

Since Y is a net, we have $\sup \operatorname{diam} C_y < \infty$, and $\inf \operatorname{vol} C_y > 0$ because Y is separated. Let

$$f(x) = \sum_{y \,:\, x \in C_y} \frac{1}{\operatorname{vol} C_y}. \tag{4.1}$$

Then f and $1/f$ are bounded a.e., so (A) provides a bi-Lipschitz homeomorphism $\phi : \mathbb{R}^n \to \mathbb{R}^n$ with Jacobian determinant f. Letting $D_y = \phi(C_y)$, we have $\operatorname{vol} \phi(D_y) = 1$.

For $z \in \mathbb{Z}^n$ let E_z denote the unit cube centered at z. Consider the relation $R \subset Y \times \mathbb{Z}^n$ given by the set of pairs (y, z) such that D_y meets E_z. Since $\operatorname{diam} D_y$ and $\operatorname{diam} E_z$ are bounded, the distance $|\phi(y) - z|$ is also bounded for all $(y, z) \in R$.

Now think of the relation R as a multi-valued map from Y to $\mathbb{Z}^n$. Then for any finite set $A \subset Y$, we have $|R(A)| \geq |A|$. Indeed, the cubes labeled by $R(A)$ cover the cells D_y labeled by A, so the inequality follows from the fact that $\operatorname{vol} D_y = \operatorname{vol} E_z = 1$. Similarly, $|R^{-1}(B)| \geq |B|$ for any finite set $B \subset \mathbb{Z}^n$.

By the transfinite form of Hall's marriage theorem [Mir, Thm. 4.2.1], R contains the graph of an injective map $\psi_1 : Y \to \mathbb{Z}^n$. Similarly, R^{-1} contains the graph of an injective map $\psi_2 : \mathbb{Z}^n \to Y$. By the Schröder-Bernstein theorem [Hal, §22], R contains the graph of a bijection $\psi : Y \to \mathbb{Z}^n$. Since $\sup |\psi(y) - \phi(y)| < \infty$, the map $\psi : Y \to \mathbb{Z}^n$ is bi-Lipschitz, proving (B). ■

The proof of (A) $\implies$ (B) shows that for any separated net Y, the quality of a bijection $\phi : Y \to \mathbb{Z}^n$ can be controlled by the quality of a solution to $\det D\phi = f$, where f is determined by the Voronoi cells as in (4.1). This fact is exploited in the next section.

5 Hölder maps

To conclude we show questions (1-3) have positive answers if we relax the Lipschitz condition to a Hölder condition.

Definition. We say $\phi : \mathbb{R}^n \to \mathbb{R}^n$ is a *homogeneous Hölder map* if there are constants $K \geq 0$ and $0 < \alpha \leq 1$ such that for $|x|, |y| \leq R$ we have

$$|\phi(x) - \phi(y)| \leq KR^{1-\alpha}|x - y|^\alpha. \tag{5.1}$$

If $\phi(x)$ satisfies (5.1), then so does $r\phi(x/r)$ for every $r > 0$; it is this sense that the Hölder condition above is homogeneous.

If ϕ and ϕ^{-1} both satisfy (5.1) then we say ϕ is a *homogeneous bi-Hölder* homeomorphism. When $\alpha = 1$ we obtain the class of bi-Lipschitz maps. Note that for any homogeneous bi-Hölder homeomorphism, we have

$$|\phi(y)| \asymp |y|$$

when $|y|$ is large. To see this, set $x = 0$ and $R = |y|$ in (5.1).

We say a map $\phi : Y \to Y'$ between subsets of $\mathbb{R}^n$ is a homogeneous bi-Hölder bijection if ϕ and ϕ^{-1} satisfy (5.1) on their respective domains.

Theorem 5.1 *Fix $n \geq 1$. Then:*

1. *For any $f \in L^\infty(\mathbb{R}^n)$ with $\inf f(x) > 0$, there is a homogeneous bi-Hölder homeomorphism $\phi : \mathbb{R}^n \to \mathbb{R}^n$ such that*
$$\mathrm{vol}(\phi(E)) = \int_E f(x)\, dx \tag{5.2}$$
for all bounded open sets $E \subset \mathbb{R}^n$.

2. *For any $f \in L^\infty(\mathbb{R}^n)$, there is a vector field v with Zygmund components such that $\operatorname{div} v = f$.*

3. *For any separated net $Y \subset \mathbb{R}^n$, there is a homogeneous bi-Hölder bijection $\psi : Y \to \mathbb{Z}^n$.*

Lemma 5.2 *Any radial function $f(r) \in L^\infty(\mathbb{R}^n)$ with $\inf f > 0$ can be realized as the Jacobian determinant of a radial bi-Lipschitz homeomorphism $\phi(r, \theta) = (\psi(r), \theta)$.*

Proof. Define $\psi : [0, \infty) \to [0, \infty)$ by
$$\frac{\psi(r)^n}{n} = \int_0^r s^{n-1} f(s)\, ds.$$
Then we have
$$\det D\phi = \frac{\psi'(r)\psi(r)^{n-1}}{r^{n-1}} = f(r).$$
The upper and lower bounds on f imply $\psi(r) \asymp r$, so by the formula above we have $\psi'(r) \asymp 1$. Thus ϕ is bi-Lipschitz. ■

Proof of Theorem 5.1.
(2). This statement follows from the general theory of singular integral operators, as sketched in §2. Note that a vector field v with Zygmund components has $|x \log x|$ modulus of continuity and generates a flow whose time-one map is Hölder [Rei2, Prop. 4].
(1). This result is due to Rivière and Ye. Consider the tiling of $\mathbb{R}^n - \{0\}$ by the dyadic annuli
$$\langle A_i = \{x \,:\, 2^i \leq |x| \leq 2^{i+1}\}, \;\; i \in \mathbb{Z}\rangle.$$

After a preliminary radial Lipschitz map, whose existence is insured by Lemma 5.2, we can assume $\int_{A_i} f = \int_{A_i} 1$ for each i. By [RY2, Thm. 2], there exists a homeomorphism $\phi_0 : A_0 \to A_0$ such that

(i) $\int_E f(x)\,dx = \mathrm{vol}(\phi_0(E))$ for any open set $E \subset A_0$;

(ii) $\phi_0(x) = x$ on ∂A_0; and

(iii) $K^{-1}|x-y|^{1/\alpha} \;\leq\; |\phi(x)-\phi(y)| \;\leq\; K|x-y|^\alpha$, where $\alpha > 0$, $K > 1$ depend only on $\|f\|_\infty + \|1/f\|_\infty$ (compare [RY2, (2.14)]).

Since A_i is simply A_0 rescaled by a factor of 2^i, we can apply this result to obtain homeomorphisms $\phi_i : A_i \to A_i$ satisfying the volume distortion equation (5.2) for $E \subset A_i$. The Hölder bounds in (iii) rescale to give the homogeneous bounds (5.1) for ϕ_i and ϕ_i^{-1}, so the ϕ_i piece together to produce the desired homogeneous bi-Hölder map $\phi : \mathbb{R}^n \to \mathbb{R}^n$.

(3). Let $Y \subset \mathbb{R}^n$ be a separated net. Let $\langle C_y \rangle$ be the Voronoi cells for Y, and let E_z denote the unit cube centered at $z \in \mathbb{Z}^n$. Define $f(x) = 1/\mathrm{vol}(C_y)$ for $x \in C_y$ as in (4.1).

By (1) there exists a homogeneous bi-Hölder map $\phi : \mathbb{R}^n \to \mathbb{R}^n$ sending $f(x)\,dx$ to the standard measure on $\mathbb{R}^n$. Letting $D_y = \phi(C_y)$ we have $\mathrm{vol}\, D_y = 1$ and $\mathrm{diam}\, D_y = O(1+|y|^{1-\alpha})$, where α is the exponent in (5.1). As in the proof of Theorem 4.1, Hall's marriage theorem provides a bijection $\psi : Y \to \mathbb{Z}^n$ such that $D_y \cap E_z \neq \emptyset$ whenever $\psi(y) = z$. Therefore

$$|\phi(y) - \psi(y)| \leq C(1+|y|^{1-\alpha}) \tag{5.3}$$

for some constant C.

We claim $\psi : Y \to \mathbb{Z}^n$ is a homogeneous bi-Hölder map. Indeed, given distinct points $x, y \in Y$ with $|x|, |y| \leq R$, by (5.1) and (5.3) we have

$$\begin{aligned} |\psi(x)-\psi(y)| &\leq |\phi(x)-\phi(y)| \;+\; |\phi(x)-\psi(x)| \;+\; |\phi(y)-\psi(y)| \\ &\leq KR^{1-\alpha}|x-y|^\alpha + 2C(1+R^{1-\alpha}) \\ &= O(R^{1-\alpha}|x-y|^\alpha) \end{aligned}$$

since $|x-y| > \epsilon > 0$ by separation of Y. This shows ψ satisfies the homogeneous Hölder condition.

To verify the same condition for ψ^{-1}, we apply the same reasoning to the inverse image cubes $F_z = \phi^{-1}(E_z)$. The Hölder condition on ϕ^{-1} gives $\mathrm{diam}(F_z) = O(1+|z|^{1-\alpha})$, and since $F_z \cap C_{\psi^{-1}(z)} \neq \emptyset$ we have

$$|\psi^{-1}(z) - \phi^{-1}(z)| \leq C'(1+|z|^{1-\alpha}).$$

Thus for distinct $z, w \in \mathbb{Z}^n$ with $|z|, |w| \leq R$ we have

$$\begin{aligned} |\psi^{-1}(z) - \psi^{-1}(w)| &\leq KR^{1-\alpha}|z-w|^\alpha + 2C'(1+R^{1-\alpha}) \\ &= O(R^{1-\alpha}|z-w|^\alpha) \end{aligned}$$

since $|z - w| \geq 1$. Therefore ψ^{-1} also satisfies (5.1) and we are done. ∎

References

[BK] D. Burago and B. Kleiner. Separated nets in Euclidean space and Jacobians of biLipschitz maps. *To appear, Geometric and Functional Analysis.*

[DM] B. Dacorogna and J. Moser. On a partial differential equation involving the Jacobian determinant. *Ann. Inst. H. Poincaré Anal. Non-linéaire* **7**(1990), 1–26.

[Gr] M. Gromov. Asymptotic invariants of infinite groups. In G. A. Niblo and M. A. Roller, editors, *Geometric Group Theory*, volume 2. Cambridge University Press, 1993.

[Hal] P. Halmos. *Naive Set Theory.* Springer-Verlag, 1974.

[Mc1] C. McMullen. *Complex Dynamics and Renormalization*, volume 135 of *Annals of Math. Studies.* Princeton University Press, 1994.

[Mc2] C. McMullen. *Renormalization and 3-Manifolds which Fiber over the Circle*, volume 142 of *Annals of Math. Studies.* Princeton University Press, 1996.

[Mir] L. Mirsky. *Transversal Theory.* Academic Press, 1971.

[Mos] J. Moser. On the volume elements on a manifold. *Trans. Amer. Math. Soc.* **120**(1965), 286–294.

[Or] D. Ornstein. A non-inequality for differential operators in the L_1 norm. *Arch. Rational Mech. Anal.* **11**(1962), 40–49.

[Rei1] H. M. Reimann. Harmonische Funktion und Jacobi- Determinanten von Diffeomorphismen. *Comment. Math. Helv.* **47**(1972), 397–408.

[Rei2] H. M. Reimann. Ordinary differential equations and quasiconformal mappings. *Invent. math.* **33**(1976), 247–270.

[RY1] T. Rivière and D. Ye. Une résolution de l'equation à forme volume prescrite. *C. R. Acad. Sci. Paris Sér. I Math.* **318**(1994), 25–28.

[RY2] T. Rivière and D. Ye. Resolutions of the prescribed volume form equation. *Nonlinear Differential Equations Appl.* **3**(1996), 323–369.

[St] E. M. Stein. *Harmonic Analysis.* Princeton University Press, 1993.

[Tol] D. Toledo. Geometric group theory, Vol. 2: Asymptotic invariants of infinite groups by M. Gromov. *Bull. Amer. Math. Soc.* **33**(1996), 395–398.

[Ye] D. Ye. Prescribing the Jacobian determinant in Sobolev spaces. *Ann. Inst. H. Poincaré Anal. Nonlinéaire* **11**(1994), 275–296.

Mathematics Department, University of California, Berkeley CA 94720-3840